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Particles and "Bumps" in Quantum Field Configurations

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We study an approach to making a precise identification of particles with "bumps" in quantum field configurations. The problem requires some effort because typical field configurations are distributions and not elements of the classical configuration space at all. We show that the part of a configuration consisting of "vacuum fluctuations" may be identified and filtered out in a very natural way, leaving a function which is in the classical configuration space. The filtered field configuration depends on the state Ψ of the field, and for an n-particle state with n particles well localized and well separated in the Newton–Wigner sense, the filtered field configuration has n bumps located where the particles are. We close with a discussion of observation in the Schrödinger representation for the free field in terms of our results.

KEY WORDS: Localization; quantum field configurations; relativistic position.

1. INTRODUCTION

While it is widely held that particles may somehow be regarded as "bumps" in a quantum field configuration, the literature appears to lack a careful study of the precise sense in which this identification may be made. The issue is too interesting to leave unattended, and so we undertake such a study in the present paper.

Our question is purely kinematical; so it is natural, and not only expeditious, to consider it in the context of free fields. However, we will restrict our attention to scalar free fields. Even in this basic context, our

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results shed at least a ray of new light on the Schrödinger picture in field theory. While the question has an intrinsic interest within orthodox quantum field theory, it is crucial in the treatment of quantum fields in terms of Nelson's stochastic mechanics. Indeed, this was our motivation. We will, however, develop the applications of our present results to stochastic mechanics in a later paper; here we restrict our attention to orthodox issues.

The essential difficulty to be confronted when attempting to identify particles with bumps in quantum field configurations is that the latter are so very rough, it is difficult to see any bumps anywhere at all. The unfortunate, familiar fact is that the typical field configuration is not a function at all, but only a distribution. Picturesquely speaking, one might say the "vacuum fluctuations" are so violent, they overwhelm the bumps, like static interference washing out a broadcast signal.

In this paper we describe a physically natural and mathematically sound way to filter out the vacuum fluctuations. This yields field configurations which are genuine functions *in the classical configuration space of the field theory*. Moreover, if the state of the quentum field is an n-particle state with n well localized and well separated particles in the Newton–Wigner sense,[6] then the filtered configuration will almost surely have n well-defined bumps in the appropriate places.

This filtering procedure takes into account the state Ψ of the field; indeed, it must for the above results to be true.

We postpone a more detailed discussion of these results until we have established sufficient notation and background pertaining to the Schrödinger picture for the Klein–Gordon field; this occupies the next section of the paper. In Section 3, we give a careful treatment of our prescription for subtracting out vacuum fluctuations, and we discuss its physical content.

We emphasize that we do not discuss the measurement procedure itself; that is, we will not give a prescription for doing an experimental observation. Here we take for granted the orthodox view that every self-adjoint operator on the state space is an observable, and that there is some experiment which will yield its singular values as results. We apply this point of view to the field operators, and ask how one could take the results of these observations and learn anything about how many particles there are and where the particles are located. Since even the particle number operator does not commute with the field operators, the answer requires at least some work.

Finally, in Section 4, we apply the prescription when the quantum field is in an n-particle state. The resulting field configurations can be computed quite explicitly, and this permits us to provide a satisfying

answer to the question raised at the outset of this investigation; we do this in the conclusion to the paper.

2. A more detailed treatment, in the same spirit and notation, of the Schrödinger picture for the free scalar field of mass m can be found in ref. 1. Here we set down just enough background to permit a careful development of our discussion.

The theory of the free scalar field of mass m is based on the Klein–Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\phi(x, t) = 0 \tag{2.1}$$

where ϕ is a real function on $\mathbb{R}^3 \times \mathbb{R}$. We interpret $\phi(\cdot, t)$ as a classical field configuration at time t, and the Klein–Gordon equation governs the classical time evolution of these configurations in exactly the same way as Newton's equation governs the evolution of configurations in an ordinary classical mechanical system.

When we quantize the field in the Schrödinger picture, the state space is the Hilbert space of square-integrable functions on the classical configuration space. Deciding which functions will constitute the classical configuration space and what measure we will put on it is closely related to choosing the sense in which we wish to solve (2.1); this choice is essentially determined by symmetry considerations. Let

$$H = (-\Delta + m^2)^{1/2}$$

and let $\mathcal{Q}$ be the completion of real $C_0^\infty(\mathbb{R}^3)$ in the Hilbertian norm

$$\|\phi\|_{\mathcal{Q}} = \|H^{1/2}\phi\|_{L^2(\mathbb{R}^3)}$$

$\mathcal{Q}$ is the *q-space* or classical configuration space of the scalar field of mass m.

To see why this choice is made for $\mathcal{Q}$, define, for any real solution ϕ of (2.1), a complex function ϕ_+ by

$$\phi_+(x, t) = \phi(x, t) + iH^{-1}\dot{\phi}(x, t)$$

Then, if we put

$$\|\phi\|_{\mathscr{H}}^2 = \|\phi(\cdot, t)\|_{\mathcal{Q}}^2 + \|H^{-1}\dot{\phi}(\cdot, t)\|_{\mathcal{Q}}^2 = \|\phi(\cdot, t)\|_{\mathcal{Q}\otimes C}^2$$

and work out the right-hand side by Fourier transforming, we see that it is independent of t and even of the Lorentz frame in which the computa-

tion is done. Therefore, the left-hand side defines a Hilbertian norm on a space of solutions of the Klein–Gordon equation. Though the solutions themselves are real, the completion is a *complex* Hilbert space in a natural way—complex numbers just multiply ϕ_+ in the usual manner. Denote this complex Hilbert space by $\mathscr{H}$. It is called the *single-particle Hilbert space*, and the Poincaré group acts on it unitarily by construction.[2]

$\mathscr{H}$ is the *phase space* of our classical field theory. Taking the real and imaginary parts of some $\phi_+(\cdot, 0)$ in $\mathscr{H}$ yields $\phi(\cdot, 0)$ and $H^{-1}\dot{\phi}(\cdot, 0)$ in $\mathscr{Q}$—the initial configuration and the initial momentum. Unfortunately, this decomposition of the phase space into configuration space and momentum space is not Poincaré invariant; it depends on the choice of the $t=0$ hyperplane.

However, whatever choice we make, we get some version of $\mathscr{Q}$ as the configuration space. It remains to put a measure μ on $\mathscr{Q}$, so that we may form the state space of square-integrable functions on $\mathscr{Q}$.

For any number of reasons, the natural choice to make for μ is the Gaussian measure on $\mathscr{Q}$ of unit covariance. For instance, this is related in an obvious way to the unit Gauss measure on $\mathscr{H}$ which is Poincaré invariant.

Here we run into a problem typical in the quantization of systems with infinitely many degrees of freedom: there is no unit Gauss measure living on $\mathscr{Q}$ itself. The reason for this is easily understood; recall that

$$\int_{\mathbb{R}^n} |x|^2 e^{-|x|^2/2} d^n x = n$$

and

$$\left[\int_{\mathbb{R}^n} (|x|^2 - n)^2 e^{-|x|^2/2} d^n x\right]^{1/2} = (2n)^{1/2}$$

This means that if we equip the n-dimensional Hilbert space $\mathbb{R}^n$ with its unit Gauss measure, this measure is more and more concentrated on the sphere of radius $\sqrt{n}$ as n increases. For an infinite-dimensional Hilbert space like $\mathscr{Q}$, we might then expect that the support of the unit Gauss measure lies outside $\mathscr{Q}$ itself, and indeed it does. It is precisely this fact which forces us to work with generalized field configurations.

In the theory of integration on Hilbert space developed by Gross,[3] one first builds a larger space on which the measure can live. One does this by taking the completion of $\mathscr{Q}$ in a smaller Banach space norm $\|\cdot\|_B$ yielding a larger Banach space B in which $\mathscr{Q}$ is densely imbedded. Gross has described the norms—there are plenty— which yield spaces capable of supporting the unit Gauss measure on $\mathscr{Q}$ in the following sense: For any

such B there is a Borel probability measure μ on B so that for each ϕ in $\mathcal{Q}$ which happens to lie in the dual of B—these are dense—the function

$$\xi \mapsto (\xi, \phi)_{\mathcal{Q}}$$

is a Gaussian random variable on the probability space $(B, \mathcal{B}, \mu)$ with mean zero and variance $\|\phi\|_{\mathcal{Q}}^2$. By an approximation argument, these random variables can be defined for every ϕ in $\mathcal{Q}$. Of course, for the reasons cited above, $\mu(\mathcal{Q}) = 0$, but in some sense B is big enough to contain the sphere of radius $\sqrt{\infty}$ in $\mathcal{Q}$, and that is where the unit Gauss measure on $\mathcal{Q}$ lives.

Making any such choice of B, set $\mathcal{K} = L^2(B, \mathcal{B}, \mu)$, the space of all *complex* square-integrable functions on B. This is the Hilbert space of states for the free scalar field of mass m. A fundamental fact discovered by Segal[4] is that $\mathcal{K}$ is naturaly isomorphic to the Fock space

$$\hat{\mathcal{K}} = \bigoplus_{n=0}^{\infty} \left(\bigotimes_{\text{sym}}^{n} \mathcal{H} \right)$$

that is, the symmetric tensor algebra over $\mathcal{H}$. Segal's result is what relates the Schrödinger picture to the more familiar Fock space picture. Moreover, since the Fock space does not involve B, Segal's result shows that any two choices of B lead to naturaly isomorphic state spaces. This partially justifies thinking of $\mathcal{K}$ as $L^2(\mathcal{Q})$.

Let $S: \mathcal{K} \to \hat{\mathcal{K}}$ denote Segal's unitary transformation. One can give an explicit representation of S; see ref. 4 or ref. 1.

Next, by construction, the time translation subgroup of the Poincaré group acts unitarily on $\mathcal{H}$, and so, acting factor by factor, it acts unitarily on Fock space $\hat{\mathcal{K}}$. Conjugating by S, it acts unitarily on $\mathcal{K}$; let $\Gamma(U(t))$ denote this unitary group. By Stone's theorem we have

$$\Gamma(U(t)) = e^{-it\mathcal{L}}$$

for some self-adjoint operator $\mathcal{L}$ on $\mathcal{K}$. This is the Hamiltonian for the free scalar field of mass m in the Schrödinger representation. The Schrödinger equation for the free field then is

$$i\frac{\partial}{\partial t}\Psi(\xi, t) = \mathcal{L}\Psi(\xi, t), \qquad \Psi(\xi, 0) = \Psi_0(\xi) \tag{2.2}$$

One can easily compute $\mathcal{L}\Psi$ and see that $\mathcal{L}$ is an infinite-dimensional ellitic partial differential operator. In fact, it is just the infinite-dimensional Ornstein–Uhlenbeck operator. All one needs to make this computation is the explicit form of S and the familiar Fock space representation of the

Hamiltonian. For details, see ref. 1. We close this section with versions of two results from that paper which we will use in the next sections. First recall that Ψ is said to be an *n-particle* state in case $S\Psi \in \bigotimes^n_{\text{sym}} \mathscr{H}$, and the zero-particle state $\Psi = 1$ is called the *vacuum state.*

Theorem 2.1. $\Psi(\xi, t)$ is a single-particle solution of (2.2) exactly when

$$\Psi(\xi, t) = (\xi, \phi(\cdot, t))_{\mathscr{L}} + i(\xi, H^{-1}\dot{\phi}(\cdot, t))_{\mathscr{L}}$$

for some solution ϕ of the Klein–Gordon equation (2.1) belonging to $\mathscr{H}$. Now, $\Psi(\xi, t)$ is an n-particle solution when it is the projection onto the n-particle subspace of $\mathscr{K}$ of a homogeneous polynomial of degree n in single-particle solutions, and the general n-particle solution is obtained as a limit of such solutions.

This result spells out the relation between solutions of the Klein–Gordon equation and the Schrödinger equation for the free field. This permits us to use the following result on the long-time asymptotics of solutions to the Klein–Gordon equation as a result on the long-time asymptotics of the Schrödinger equation.

Theorem 2.2. Let $\phi(x, t)$ be any solution of the Klein–Gordon equation (2.1) with ϕ_+ belonging to $\mathscr{H}$. Then

$$\begin{gathered} \lim_{t\to\infty} \|\phi(\cdot, t)\|^2_{\mathscr{L}} = \lim_{t\to\infty} \|H^{-1}\dot{\phi}(\cdot, t)\|^2_{\mathscr{L}} = \|\phi_+(\cdot, 0)\|^2_{\mathscr{H}}/2 \\ \lim_{t\to\infty} (\phi(\cdot, t), H^{-1}\dot{\phi}(\cdot, t))_{\mathscr{L}} = 0 \end{gathered} \tag{2.3}$$

Moreover, if $\phi(\cdot, 0)$ and $H^{-1}\dot{\phi}(\cdot, 0)$ belong to the Schwartz space $\mathscr{S}$, the convergence is faster than any inverse power of t.

Indeed, for any $v \in \mathbb{R}^3$, $|v| < 1$, and any f which is the Fourier transform of a bounded function supported in the ball of radius $2m|v|/(1-v^2)^{1/2}$,

$$\begin{gathered} \lim_{t\to\infty} \int f(x - vt)[|H^{1/2}\phi(x, t)|^2 - |H^{-1/2}\dot{\phi}(x, t)|^2]\, d^3x = 0 \\ \lim_{t\to\infty} \int f(x - vt)[H^{1/2}\phi(x, t)\, H^{-1/2}\dot{\phi}(x, t)]\, d^3x = 0 \end{gathered} \tag{2.4}$$

faster than any power of t when $\phi(\cdot, 0)$ and $H^{-1}\dot{\phi}(\cdot, 0)$ belong to $\mathscr{S}$. If also $mv/(1-v^2)^{1/2}$ belongs to the support of $\hat{\phi}_+(\cdot, 0)$ and 0 belongs to the support of $\hat{f}$, then

$$\lim_{t\to\infty} t^3 \int f(x - vt)|H^{1/2}\phi_+(x, t)|^2\, d^3x \tag{2.5}$$

exists and is not zero.

Proof. The first part is a limiting case of the second, and it is discussed in detail in ref. 1. Therefore we limit ourselves to a sketch of the simple stationary phase argument which yields the second part.

Let us introduce $\psi_+ = H^{1/2}\phi_+$, which is a normalized element of $L^2(\mathbb{R}^n)$. This appears frequently in our computations, and it is what tends to a solution of the free Schrödinger equation (after adjusting the phase) in the nonrelativistic limit.

Fourier transforming, we obtain

$$\begin{aligned}
&\int f(x-vt)[\psi_+(x,t)]^2\,d^3x \\
&\quad = \frac{1}{(2\pi)^3}\iint \{\exp[it(k'^2+m^2)^{1/2}+(k^2+m^2)^{1/2}+v\cdot(k-k')]\} \\
&\qquad \times \hat{f}(k-k')\,\hat{\psi}_+(k',0)\,\hat{\psi}_+(-k,0)\,d^3k\,d^3k' \\
&\quad \equiv \frac{1}{(2\pi)^3}\iint \{\exp[itF_v(k,k')]\}\,\Phi(k,k')\,d^3k\,d^3k' \qquad (2.6)
\end{aligned}$$

where $F_v(k,k') = (k'^2+m^2)^{1/2}+(k^2+m^2)^{1/2}+v\cdot(k-k')$.

F_v possesses just one critical point, namely $k = mv/(1-v^2)^{1/2}$, $k' = -mv/(1-v^2)^{1/2}$. By our hypothesis on f and v, this critical point lies outside the support of Φ. Since $\hat{\psi}_+(\cdot,0)$ belongs to $\mathscr{S}$, a standard stationary phase result[8,9] says that the integral (1.3) vanishes to all orders in t.

Taking the real and imaginary parts of this yields (2.4).

On the other hand, reexpressing (2.5) using the Fourier transform yields an oscillatory integral with the unique critical point of the phase function being $k = mv/(1-v^2)^{1/2}$, $k' = mv/(1-v^2)^{1/2}$. Under the addition hypothesis on the supports of $\hat{f}$ and $\hat{\psi}_+$, the critical point belongs to the support of the integral, and by the standard argument in refs. 8 and 9 the integral decays like t^{-3} (since it is a six-dimensional integral). ∎

3. As we have seen in the last section, it is necessary to use wave functions Ψ which are functions not on the classical configuration space $\mathscr{Q}$—as in the quantization of a system with finitely many degrees of freedom—but on a larger space B of *generalized configurations.* It may be shown that even in one spatial dimension, B must be taken to consist of distributions; any function space is too small.[5] For any choice of B, $\mu(\mathscr{Q}) = 0$.

This, then, is our problem: when we observe the configuration of our system—however this is to be actually done in the laboratory—we are supposed to get a classical field configuration as a result. The probability

that we observe a configuration in some particular set of the classical configuration space is given by integrating the squared modulus of the wave function for the state of our system over the set in question. That at least is the orthodox Born interpretation of the wave function's physical content. Unfortunately, the *entire* classical field configuration space is a set of measure zero in the case at hand. If a configuration ξ is selected at random according to the law $|\Psi(\xi)|^2 \mu(d\xi)$, the result will not even be a function. The typical ξ will have wild behavior everywhere with no recognizable bumps. Moreover, this is still true after smearing—although then at least we would have a function. We wish to reconcile this wild behavior with the orthodox Born interpretation and with the folklore picture of bumps in the field representing particles.

We claim that this wild behavior is entirely due to "vacuum fluctuations" and that these may be "filtered out" in a natural fashion. Moreover, for finite-energy states, the filtered configuration is a classical field configuration—an element of $\mathcal{Q}$—with bumps in the right places.

We start to substantiate these claims by describing the filtering prescription. Fix a wave function Ψ. For the sake of a simple notation let $\Phi(f)$ denote the random variable

$$\Phi(f) = (\cdot, f)_{\mathcal{Q}}$$

on $(B, \mathcal{B}, |\Psi|^2 \, d\mu)$ for any $f \in \mathcal{Q}$. We will usually write $\Phi(\xi, f)$ in place of $(\Phi(f))(\xi)$. Also, we will write E^{Ψ} to denote expectation with respect to the probability law $|\Psi|^2 \, d\mu$. Before carefully describing the filtering procedure in three spatial dimensions, we give a heuristic description in one spatial dimension, adapting all our notation in the obvious way.

For each x in $\mathbb{R}$, define the random variable

$$\beta(x) = \Phi(H^{-1/2} 1_{(0,x)})$$

Since $\|H^{-1/2} 1_{(0,x)}\|_{\mathcal{Q}} = |x|^{1/2}$, in the vacuum state (that is, $\Psi = 1$), the probability law is just μ; each $\beta(x)$ is Gaussian with mean zero and variance $|x|$. Making a similar computation of the covariances, one sees that $x \mapsto \beta(x)$ is a two-sided standard Wiener process with $\beta(0) = 0$. The weak derivative $(d/dx)\beta(x)$ is then the white noise process. Formally regarding the elements ξ of B as functions, we may write

$$\Phi(\xi, f) = \int_{\mathbb{R}} H\xi(y) \, f(y) \, dy \tag{3.1}$$

Then $\beta(x) = \int_{\mathbb{R}} H^{1/2} \xi(y) \, 1_{(0,x)}(y) \, dy$ and since $(d/dy) \, 1_{(0,x)}(y) = \delta(x - y)$, we have

$$\frac{d}{dx} \beta(x) = H^{1/2} \xi(x)$$

That is, in the vacuum state, $H^{1/2}\xi(x)$ is just the white noise process. The fact that the fluctuations of $H^{1/2}\xi(x)$, unlike those of $\xi(x)$ itself, are independent for x in disjoint regions under the vacuum state law is what will enable us to filter out the vacuum fluctuations in a physically meaningful way. Now let Ψ be some other state, and consider the random variables $\beta(x)$ with respect to the probability law $|\Psi(\xi)|^2\,\mu(d\xi)$. Then $x \mapsto \beta(x)$ is no longer a Wiener process, and so its derivative is no longer the white noise process. Rather, we will have something like

$$\frac{d}{dx}\beta(x) = \eta(x) + \text{white noise} \tag{3.2}$$

Once we have made sense of this equation, we will interpret $\eta(x)$ as $H^{1/2}\xi(x)$ filtered of the "vacuum fluctuations" in the state Ψ, and we will interpret the white noise as "vacuum fluctuations."

To produce a meaningful version of (3.2) in three spatial dimensions, as well as to explain the physical content of our filtering prescription, we introduce the σ-algebras

$$\sigma(\Lambda) = \sigma\{\Phi(f) \mid \text{support}(H^{-1/2}f) \subset \Lambda\}$$

for all open subsets Λ of $\mathbb{R}^3$. Let $C_h(x)$ be the closed cube in $\mathbb{R}^3$ with sides of length h parallel to the (fixed) coordinate axes and center $x \in \mathbb{R}^3$. Let $D_h(x)$ denote its complement.

Definition 3.1. In the state Ψ, the *filtering of the field configuration* is the map given by $\xi \mapsto \eta(x, \xi)$ where

$$\eta^{\Psi}(x, \xi) = \lim_{h \to 0} \frac{1}{h^3} E^{\Psi}\{\Phi(H^{-1/2}1_{C_h(x)}) \mid \sigma(D_h(x))\} \tag{3.3}$$

provided this limit exists in probability with respect to $|\Psi(\xi)|^2\,\mu(d\xi)$.

The conditional expectation in (3.3) eliminates the white noise in (3.2); and taking the limit in (3.3) amounts, in one spatial dimension, to taking the derivative in (3.2).

We now wish to explain why this is a natural definition. Consider the following straightforward approach to the problem of measuring the field configuration in the state Ψ: Pick a small $h > 0$ and a large integer N. For each $\mathbf{n} \in [-N, N]^3$ define

$$f_{\mathbf{n}} = h^{-3/2}H^{-1/2}1_{C_h(h\mathbf{n})}$$

Then the $\{f_{\mathbf{n}} \mid \mathbf{n} \in [-N, N]^3\}$ are an orthonormal system in $\mathcal{Q}$. While it is incomplete, the projection onto its span tends strongly to the identity as h

decreases to zero and N increases to infinity. As this happens, the associated system of commuting observables

$$\{\Phi(f_{\mathbf{n}}) \mid \mathbf{n} \in [-N, N]^3\}$$

tends to a complete system of commuting observables.

Now repeatedly prepare the system to be in the state Ψ and measure the observables above, obtaining values $\alpha_{\mathbf{n}}^{(k)}$, where the k denotes the kth trial. After a very large number of trials, one is able to deduce the value of the conditional expectations

$$E^{\Psi}\{\Phi(f_{\mathbf{n}}) \mid \Phi(f_{\mathbf{m}}) = \alpha_{\mathbf{m}} \quad \text{for } \mathbf{m} \neq \mathbf{n}\}$$

for any given specification of the outcomes α_n.

In the vacuum state each of the random variables $\Phi(f_n)$ is a unit Gaussian *independent* of the others. Actually, since

$$h^{-3} \int_{C_h(x)} H^{1/2}\xi(y)\, dy = h^{-3}\Phi(H^{-1/2}1_{C_h(x)})$$

it is $h^{-3/2}\Phi(f_n)$ that concerns us, and this has variance h^{-3}.

In a generic state Ψ, the random variables $h^{-3/2}\Phi(f_n)$ will no longer be Gaussian, but they will still have a variance which diverges as h decreases to zero. And though they will no longer be mutually independent, even the conditional variance—given the outcomes for all the other observables—will still diverges as h decreases to zero. The only way to get a definite value for

$$h^{-3} \int_{C_h(h\mathbf{n})} H^{1/2}\xi(x)\, dx$$

in the state Ψ is to take those trials of our experiment for which the observed value of $\Phi(f_m)$ is very close to $(\xi, f_m)_{\mathcal{Q}}$ for all $m \neq n$—so we know the observation is yielding a generalized configuration close to ξ—and then averaging over the observed values of $\Phi(f_n)$ in these trials. Idealizing this by taking h to zero and N to infinity yields the definition given above of the filtered field configuration at ξ in the state Ψ.

The procedure of filtering described above parallels the quantum mechanical prescription for measurement of an observable with continuous spectrum.

Notice that $\eta(\cdot, \xi)$ is $H^{1/2}\xi$, and not ξ itself, filtered of vacuum fluctuations. One could make a case for referring to $H^{-1/2}\eta(\cdot, \xi)$ as the filtered field configuration; however, as we have just argued, it is $\eta(\cdot, \xi)$ that is directly accessable to observation—again granted the orthodox assumptions about observables and measurements.

Clearly, in the vacuum state, this filtered field configuration vanishes identically on account of the independence discussed above. This is as it should be; in the vacuum state one has nothing but vacuum fluctuations. In the next section we show that for states of definite particle number, the filtered field configuration not only exists, it is quite well behaved.

4. Consider first the case where Ψ is a one-particle state. By Theorem 3.1, Ψ has the form

$$\Psi(\xi) = (\xi, \phi_+)_{\mathscr{Q}}$$

where $\phi_+ = \phi + iH^{-1}\dot{\phi}$ with $\|\phi_+\|_{\mathscr{H}} = 1$. Extending Φ to be complex linear, we can rewrite this using the notation of the last section as

$$\Psi_t(\xi) = (\Phi(\phi_+(\cdot, t))(\xi)$$

For any measurable subset $A \subset \mathbb{R}^3$, let P_A be the orthogonal projection (in $\mathscr{Q}$ as well as $\mathscr{H}$)

$$P_A = H^{-1/2} 1_A H^{1/2}$$

Clearly, if $A \cap B = \varnothing$, $P_A \perp P_B$. Write

$$\phi_+ = \phi_{+,h} + \phi_+^h$$

where $\phi_{+,h} = P_{C_h(x)}\phi_+$ for some fixed x. With this notation in hand, we may now easily compute the filtered field configuration.

Note that

$$|\Psi(\xi)|^2 = |\Phi(\phi_{+,h})|^2 + |\Phi(\phi_+^h)|^2 + 2 \operatorname{Re} \Phi(\phi_{+,h}^*)\, \Phi(\phi_+^h)$$

Then if F is any bounded $\sigma(D_h(x))$ measurable random variable,

$$\begin{aligned}
&E^{\Psi}(\Phi(H^{-1/2}1_{C_h(x)})F) \\
&\quad = E^1(|\Psi|^2\, \Phi(H^{-1/2}1_{C_h(x)})F) \\
&\quad = E^1(|\Phi(\phi_{+,h})|^2 \Phi(H^{-1/2}1_{C_h(x)}))\, E^1(F) \\
&\qquad + E^1(\Phi(H^{-1/2}1_{C_h(x)}))\, E^1(|\Phi(\phi_+^h)|^2 F) \\
&\qquad + 2 \operatorname{Re} E^1(\Phi(\phi_{+,h}^*)\, \Phi(H^{-1/2}1_{C_h(x)}))\, E^1(\Phi(\phi_+^h)F)
\end{aligned}$$

by independence.

Obviously,

$$E^1(\Phi(H^{-1/2}1_{C_H(x)})) = 0$$

Writing

$$E^1(|\Phi(\phi_{+,h})|^2\,\Phi(H^{-1/2}1_{C_h(x)}))$$

out as a two-dimensional Gaussian integral, one sees that it, too, vanishes. Finally,

$$E^1(\Phi(\phi^*_{+,h})\,\Phi(H^{-1/2}1_{C_h(x)})) = \int_{C_h(x)} H^{+1/2}\phi^*_{+,h}(y)\,dy$$

We therefore have

$$E^\Psi(\Phi(H^{-1/2}1_{C_h(x)})F) = 2\,\mathrm{Re}\left[\int_{C_h(x)} H^{1/2}\phi_{+,h}(y)\,dy\right] E^\Psi(|\Psi|^{-2}\Phi(\phi^h_+)F)$$

so that, since F is arbitrary,

$$h^{-3}E^\Psi\{\Phi(H^{-1/2}1_{C_h(x)})\,|\,O(D_h(x))\}$$
$$= 2\,\mathrm{Re}\left[h^{-3}\int_{C_h(x)} H^{1/2}\phi_{+,h}(y)\,dy\right] E^\Psi(\{|\Psi|^{-2}\Phi(\phi^h_+)\,|\,O(D_h(x)\} \quad (4.1)$$

But as h decreases to zero, ϕ^h_+ tends strongly to ϕ_+, and $O(D_h(x))$ increases to $B(B)$. Therefore, the conditional expectation on the right just above tends almost surely to $|\Psi|^{-2}\Phi(\phi_+)$, which we define to be zero on the null set where Ψ vanishes. Also,

$$\lim_{h\to 0} h^{-3}\int_{C_h(x)} H^{1/2}\phi^*_+(y)\,dy = H^{1/2}\phi^*_+(x)$$

for almost every x.

This proves the following result.

Theorem 4.1. For the single-particle state $\Psi_t(\xi) = \Phi(\phi_+(\cdot, t))(\xi)$, the filtered field configuration $\eta_t(x, \xi)$ exists and is given by

$$\eta^\Psi_t(x, \xi) = 2\,|\Psi(\xi)|^{-2}\,\mathrm{Re}[(\Phi(\phi_+(\cdot, t))(\xi)\,H^{1/2}\phi^*_+(x, t)] \quad (4.2)$$

where the right-hand side is defined to be zero where Ψ vanishes.

Note that $\eta(x, \xi)$ is a linear combination of $H^{1/2}\phi(x)$ and $H^{-1/2}\dot{\phi}(x)$, so $\eta(\cdot, \xi)$ is always an element of $L^2(\mathbb{R}^3)$. Moreover, when Ψ is in the form domain of H—which is the case exactly when ϕ_+ is in the form domain of H in $\mathscr{H}^{(1)}$—$\eta(\cdot, \xi)$ is actually an element of the classical field configuration space $\mathscr{Q}$. That is, in a finite-energy single-particle state, the filtered field configuration belongs to the classical field configuration space.

It is perfectly natural that the filtered field configuration is only in $\mathscr{Q}$ when Ψ is a finite-energy state—as we have pointed out before, η is $H^{1/2}\xi$ and not ξ itself filtered of vacuum fluctuations. The $H^{-1/2}\eta(\cdot, \xi)$ is always in $\mathscr{Q}$. Similar conclusions apply to $H^{-1/2}\eta$, though the nonlocal operator $H^{-1/2}$ will smear the bumps a bit (very little if the energy is close to the rest energy). However, for the reasons we explained earlier, we prefer to work with $\eta(\cdot, \xi)$.

We are now in a position to look for bumps; we take a fixed single-particle state Ψ as above. Writing out (4.2) a bit more explicitly, the filtered configuration $\eta(\cdot, \xi)$ is given by

$$\eta^{\Psi}(x, \xi) = 2|\Psi|^{-2}\{(\Phi(\phi)\xi)(H^{1/2}\phi(x)) + (\Phi(H^{-1}\dot{\phi})\xi)(H^{-1/2}\dot{\phi}(x))\} \quad (4.3)$$

Recall that to say $\phi_+ \in \mathscr{H}$ is well localized in the Newton–Wigner sense[6,7] near $x_0 \in \mathbb{R}^3$ is to say that the functions $H^{1/2}\phi$ and $H^{-1/2}\dot{\phi}$ are well localized near x_0. Clearly, then, $\eta(\cdot, \xi)$ is well localized near x_0 exactly when ϕ_+ is well localized near x_0 in the Newton–Wigner sense—whatever specific interpretation is made of "well localized." Of course, the localized distrubance might well have many oscillations—so one should not picture a simple, smooth bump—but bearing this in mind, it will do no harm to refer to it as a bump.

Note that for any given ξ and t, the support of $\eta_t(\cdot, \xi)$ gives only partial information about the location of the particles in the Newton–Wigner sense. The support of $\eta_t(\cdot, \xi)$ is contained in the union of the supports of $H^{1/2}\phi$ and $H^{-1/2}\dot{\phi}$—that is, the support of ϕ_+. But the possible cancellations may leave the containment strict. The partial information is such that given knowledge of $\eta_t(\cdot, \xi)$, one cannot compute $\eta_t(\cdot, \xi)$ for later t.

This said, we turn to a further examination of the sense in which we can say "the bump is where the particle is."

Fix a one-particle state Ψ_t and let η_t be the corresponding filtered field configuration. Then for any $v \in \mathbb{R}^3$, $|v| < 1$, and any bounded function f on $\mathbb{R}^3$ define

$$X_t(\xi) = \frac{\int f(x - vt)|\eta_t(x, \xi)|^2 \, d^3x}{\int |\eta_t(x, \xi)|^2 \, d^3x}$$

If f is the characteristic function of a set A and we put $A_t \equiv \{x \mid (x - vt) \in A\}$, then $X_t(\xi)$ represents the fraction of the bump contained in A_t as seen by the measurement procedure considered here.

On the other hand, when f and v satisfy the hypotheses of Theorem 2.2, we will see that X_t has simple asymptotics for large t. While this is never the case when f is a characteristic function, as long as the dimensions of the set are not too small compared to mv, 1_A will be close,

in L^2 say, to a function satisfying the hypothesis of Theorem 2.2. [Indeed, if the set A is too small, the uncertainty principle prevents us from making a good approximation in the class we consider. If the product of $mv/(1-v^2)^{1/2}$ with a characteristic linear dimension of A is large compared to 1, then we can make a good approximation. In the case of one spatial dimension, a simple computation with A an interval clarifies this argument.]

Theorem 4.2. Let Ψ_t be the single-particle solution of the free field Schrödinger equation corresponding to a solution of the Klein–Gordon equation with $\phi_+(\cdot, 0) \in \mathscr{S}$. Then with f and v as in Theorem 2.2 and X_t as above,

$$\lim_{t \to \infty} t^3 |X_t(\xi) - Y_t| = 0$$

where

$$Y_t = \int f(x - vt) |H^{1/2} \phi_+(x, t)|^2 \, d^3x$$

Remark. $\int 1_A(x - vt) |H^{1/2} \phi_+(x, t)|^2 \, d^3x$ is just the Newton–Wigner probability of finding a particle in A_t in the single-particle state $\phi_+(\cdot, t)$. Insofar as we can approximate 1_A by a function satisfying the hypothesis of Theorem 2.2, i.e., when A is not too small, the theorem says the bump is where the Newton–Wigner localization theory says the particle is.

Proof. By (4.3),

$$\begin{aligned} &\int_{\mathbb{R}^3} |\eta_t^{\Psi}(x, \cdot)|^2 \, dx \\ &\quad = 4 \{ \Phi(\phi_t)^2 \, \|\phi_t\|_2^2 + \Phi(H^{-1}\dot{\phi}_t)^2 \, \|H^{-1}\dot{\phi}_t\|_2^2 \\ &\qquad + 2\Phi(\phi_t) \, \Phi(H^{-1}\dot{\phi}_t)(\phi_t, H^{-1}\dot{\phi}_t)_2 \} \, |\Psi|^{-4} \end{aligned}$$

Then, by Theorem 2.2,

$$\int_{\mathbb{R}^3} |\eta_t^{\Psi}(x, \cdot)|^2 \, d^3x = 2 |\Psi_t|^2 \, [1 + o(t^{-\infty})]$$

and the cross terms above rapidly vanish, and $\|\phi_t\|_2^2, \, \|H^{-1}\dot{\phi}_t\|_2^2 \to 1/2$.

A similar application of Theorem 2.2 to the numerator shows that

$$\lim_{t \to \infty} t^3 \left| \int f(x - vt) |\eta_t^{\Psi}(x, \cdot)|^2 \, d^3x - 2 |\Psi_t|^2 \int f(x - vt) |H^{1/2} \phi_+(t)|^2 \, d^3x \right| = 0$$

Together, these give the result. ∎

Notice that Y_t is independent of ξ; X_t becomes deterministic for large times.

The case of several particles can be handled in very much the same way. Consider the case of greatest interest; that is, consider a state Ψ which is an n-particle state with the n particles well localized and well separated in the Newton–Wigner sense. More explicitly, let $\{\phi_+^{(1)},..., \phi_+^{(n)}\} \subset \mathscr{H}$ be a set of n normalized, well-localized, and well-separated elements of $\mathscr{H}$. Insofar as these are well separated and well localized, they are orthonormal. For convenience, let us suppose they are exactly orthonormal. Then the corresponding n-particle state Ψ is simply given by

$$\Psi(\xi) = \prod_{j=1}^{n} \Phi(\phi_+^{(j)}), \xi) \tag{4.4}$$

and so Ψ is the product of η independent random variables. This makes it easy to extend the computation of η in the one-particle case to the present context. For x near the support of $\phi_+^{(j)}$, only the $\Phi(\phi_+^{(j)})^2$ factor of $|\Psi|^2$ enters the computation, and it enters just as it would in the one-particle case.

One sees as before that the filtered field configuration exists, belongs to $\mathscr{Q}$ provided Ψ is a finite-energy state, and possesses n well-defined, well-localized bumps in the place where the n particles are supposed to be.

The converse is not so simple. If one filters in the state Ψ and finds n well-defined bumps, this does not mean Ψ is an n-particle state. For example, two bumps could arise from a genuine two-particle state or from a superposition of one-particle states.

A difference between one- and many-particle states can already be seen by looking at the filtered configuration. One has for example

Theorem 4.3. If $\Psi^{(2)}$ is a two-particle state constructed from two elements ϕ_+^1 and ϕ_+^2 of $\mathscr{H}$, then

$$\eta_t^{\Psi^{(2)}}(x, \xi) = \begin{cases} |\Psi^{(2)}(\xi)|^{-2} \operatorname{Re}([H^{1/2}\phi_+^{*1}(x, t) \cdot \Phi(\phi_+^2)(\xi) \\ \qquad + H^{1/2}\phi_+^{*2}(x, t)\, \Phi(\phi_+^1)(\xi)]\, \Psi^{(2)}(\xi)) \\ 0 \quad \text{if} \quad \Psi^{(2)}(\xi) = 0 \end{cases}$$

Notice that $\eta_t^{\Psi^{(2)}}(x, \xi)$ is of the form

$$\eta_t^{\Psi^{(2)}}(x, \xi) = a_1(\xi)\, H^{1/2}\phi^1(x) + a_2(\xi)\, H^{1/2}\phi^2(x) + b_1(\xi)\, H^{-1/2}\dot{\phi}^1(x) + b_2(\xi)\, H^{-1/2}\dot{\phi}^2(x)$$

where a_1, b_1 are random variables.

From Theorem 4.1 it follows instead that, if $\Psi^{(1)}$ is a one-particle state constructed from the element $\alpha\phi^1_+ + \beta\phi^2_+$, then $\eta_t^{\Psi^{(1)}}(x, \xi)$ is of the form

$$\eta_t^{\Psi^{(1)}}(x, \xi) = c(\xi)\, H^{1/2}(\alpha\phi^1_+(x) + \beta\phi^2_+(x) + d(\xi)\, H^{-1/2}(\alpha\dot{\phi}^1_+(x) + \beta\dot{\phi}^2_+(x)$$

where c, d are random variables.

Another natural procedure to decide whether bumps correspond to n particles consists in introducing the filtering of "m-point configurations" (outcome of measurements in which the value of the field is measured in m different points).

Definition. The filtering of the m-point configuration in the state Ψ is the map $\xi \to \eta^{\Psi}_{(m)}(x_1 \cdots x_m; \xi)$ defined by

$$\eta^{\Psi}_{(m)}(x_1 \cdots x_m; \xi) \equiv \lim_{h \to 0} \frac{1}{h^{3m}} E^{\Psi}\{\phi(H^{-1/2} 1_{c_h(x_1)})\, \phi(H^{-1/2} 1_{c_h(x_m)}) \,|\, O(D_h(\mathbf{x}))\}$$

provided this limit exists in probability.

Here $D_h(\mathbf{x})$ is the complement in R^3 of the union of the closed cubes $C_h(x_i)$, $i = 1, \ldots, m$.

One can verify that for every m, if Ψ is an n-particle state, $\eta^{\Psi}_{(m)}$ exists and belongs to $Q \otimes \cdots \otimes Q$ provided Ψ is a finite-energy state. The filtering of the m-point configuration in the state Ψ distinguishes between states of different particle number in the following way. Define the m-point correlations for the field configuration in a state Ψ as the functions $\zeta^{\Psi}_{(\mathbf{m})}$ such that

$$\eta^{\Psi}_{(n)}(x_1 \cdots x_n, \xi) = \sum_{\pi} \zeta^{\Psi}_{(m_1)} \cdots \zeta^{\Psi}_{(m_K)}$$

where $(m_1, \ldots, m_K)$ is a partition of $x_1 \cdots x_n$ in subsets containing $m_1, \ldots, m_k$ elements, $m_1 + \cdots + m_k = n$, not counting permutations within each subset.

One has then

Theorem 4.5. For every n, $\zeta^{\Psi}_{(m)}(x_1 \cdots x_m; \xi) = 0$ for $m \geqslant 2n$ precisely if Ψ is a state with less than n particles.

In terms of the correlation functions one can also describe the asymptotic decoupling under the free evolution, when $t \to \pm\infty$, of the m-point configurations for n-particle states constructed from elements $\phi^1_+, \ldots, \phi^{\eta}_+$ of $\mathscr{H}$ which have disjoint support in momentum space. The precise results are of the type of Theorem 4.2; we shall omit here the details.

5. CONCLUSION

We have seen that while the typical field configuration in the support of a wave function Ψ for the free Klein–Gordon field theory is a distribution and not an element of the classical field configuration space, this is simply due to the effect of "vacuum fluctuations" and these may be filtered out in a physically natural, mathematically meaningful fashion.

What is left after the filtering is done is an element of the classical field configuration space, provided the field is in a finite-energy state. Moreover, if the state is an n-particle state with n well-localized and well-separated particles in the Newton–Wigner sense, then the filtered field configuration almost surely has n well-localized bumps at the particle positions.

This last result permits a quite satisfactory identification of particles with bumps in quantum field configurations, and so justifies some related folklore, as well as shedding some light on the Born interpretation for the wave function in quantum field theory.

Moreover, the filtered field configuration provides a quite natural means of discussing the localization of states in quantum field theory. The notion of localization this leads to is in good agreement with the Newton–Wigner notion of localization when the latter can be applied—the Newton–Wigner theory is, strictly speaking, a theory of localization in the single-particle Hilbert space $\mathscr{H}$, and not a field-theoretic theory.

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